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A Gradient Flow Approach to Optimal Model Reduction of Discrete-Time Periodic Systems

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Abstract. This paper is concerned with the optimal model reduction for linear discrete periodic time-varying systems and digital filters. Specifically, for a given stable periodic time-varying model, we shall seek a lower order periodic time-varying model to approximate the original model in an optimal H_2 norm sense. By orthogonal projections of the original model, we convert the optimal periodic model reduction problem into an unconstrained optimization problem. Two effective algorithms are then developed to solve the optimization problem. The algorithms ensure that the H_2 cost decreases monotonically and converges to an optimal (local) solution. Numerical examples are given to demonstrate the computational efficiency of the proposed method. The present paper extends the optimal model reduction for linear time invariant systems to linear periodic discrete time-varying systems.

Key words: Digital filters, Discrete time systems, Model reduction, H_2 optimization, Periodic systems and filters

1. Introduction

Cyclostationary processes arise in many different fields ranging from economics and management, to biology, communications, signal processing and control of multirate plants [13, 14, 16, 18, 22, 24, 31]. In communications and signal processing [4], cyclostationary characteristics appear, for example, in bauded data transmission, amplitude-modulated signals and video signals. Naturally, periodic models play a key role for analysis, design and simulation of cyclostationary processes. Furthermore, periodic filters and controllers have been used to improve the control and filtering performance for time-invariant systems [5, 14]. On the other hand, due to the complexity of engineering systems as well as the requirement for better control and filtering performance, quite often we are faced with high order periodic models which are undesirable due to the difficulty in analysis, computational inefficiency and high implementation cost. Therefore, the problem of model reduction for periodic systems is of considerable interest and practical importance.

The model reduction problem has been extensively investigated for linear time invariant systems and a number of results have been presented. Most of the existing work is for continuous-time models and there are basically two approaches to the model reduction, namely the error-bound based approach and optimality-based approach. A well known error-bound based approach is the so-called balanced truncation proposed by Moore [20], (see, also, [3, 8, 21], [25], and the references therein). Recently, the H_∞ and H_2 model reduction using linear matrix inequalities has also been studied in [6, 10]. For the optimality-based approach, work has been concerned with minimizing the L_2-norm of the discrepancy between the original model and a reduced one, (see, e.g., [11, 12, 26], [27]). These works basically derive first order necessary conditions for optimality but have difficulties in efficiently computing the lower order optimal model. In addtion, [23] proposed an order reduction algorithm and established its convergence for single-input single-output system.

The optimal model reduction under the H_2 specification is to minimize the root mean square value of the model reduction error over the entire frequency spectrum. An algorithm was recently presented in [30] for computing a continuous time locally optimal H_2 reduced model, where it is indicated that whether a global minimum of the H_2 model reduction optimization exists is still unclear. Although the answer to this question for the discrete time systems is positive [2], the optimization problem is known to be non-convex and is equivalent to a reduced order output feedback optimal control problem. To our knowledge, this is still an open problem and there lacks of global optimization method for analytically solving or efficiently computing the problem in the general multi-input and multi-output case.

In the context of periodic systems, the problem of H_2 model reduction becomes much harder due to the difficulties caused by the time-varying dynamics. Although there exist methods to convert periodic systems into time-invariant models [14, 17, 18], they are not directly applicable to the model reduction problem because of the severe causality constraint and difficulties in representing the time-varying state dimension and converting the time-invariant model back into the periodic realization. It is worth noting that there is no guarantee on the order of periodic realization of the obtained lower order time-invariant model even if the time-invariant model can be converted into an equivalent periodic one. Therefore, there is little result known in periodic system model reduction.

In this paper, we consider the model reduction problem for linear periodic discrete time systems and digital filters under the H_2 performance specification. By projecting the original periodic model using orthogonal matrices and applying the well known lifting technique for periodic systems [14, 18], the H_2 optimal periodic model reduction problem is formulated as an unconstrained optimization problem. Based on the gradient flow of the cost function and the first order condition for minimality of calculus, two algorithms for computing the H_2 optimal reduced order model are given. These algorithms are simple in computation and asymptotically converge to a local optimal solution. In addition, the algorithms allow the state dimension of the reduced order model to be specified to be time-varying. Given that there has been no global results for the H_2 optimal model reduction for both linear time-invariant and periodic systems available, our local result makes an original contribution to this area.

This paper is organized as follows. Section 2 formulates the order reduction problem for discrete time periodic systems with time-varying order. Section 3 derives the H_2 cost function and its gradient for the optimal order reduction. A continuous time algorithm and a discrete iterative algorithm with convergence analysis are presented in Section 4 for the H_2 optimal order reduction. In Section 5, two examples are presented with numerical computation and simulation to demonstrate the effectiveness of the proposed model reduction method.

2. Problem formulation

2.1. LINEAR PERIODIC SYSTEMS

Consider an N-periodic discrete-time system $y = \Sigma u$ in the state space equation form:

$$(\Sigma): \begin{aligned} x_{k+1} &= A_k x_k + B_k u_k \qquad (2.1)\\ y_k &= C_k x_k + D_k u_k \qquad (2.2)\end{aligned}$$

where $x_k \in \mathcal{R}^{n_k}$ is the state with time-varying dimension n_k, $u_k \in \mathcal{R}^m$ is the input and $y_k \in \mathcal{R}^p$ is the output of the system, $A_k \in \mathcal{R}^{n_{k+1}\times n_k}$, $B_k \in \mathcal{R}^{n_{k+1}\times m}$, $C_k \in \mathcal{R}^{p\times n_k}$ and $D_k \in \mathcal{R}^{p\times m}$ are N-periodic matrices of the system satisfying

$$A_{k+N} = A_k, \quad B_{k+N} = B_k, \quad C_{k+N} = C_k, \quad D_{k+N} = D_k \tag{2.3}$$

Let the transition matrix of the N-periodic system Σ be

$$\Phi_{i,j} = \begin{cases} A_{i-1}A_{i-2}\cdots A_j, & i > j \\ I, & i = j \end{cases} \tag{2.4}$$

It is well known that the periodic system Σ is stable if and only if all the eigenvalues of $\Phi_{i+N,i}$ are within the unit circle of the complex plane for all $0 \leqslant i \leqslant N-1$.

Let $\|\cdot\|_2$ denote the H_2 norm of a discrete time system. We define the H_2 norm of the stable N-periodic system Σ as

$$\|\Sigma\|_2 = \sqrt{\frac{1}{N}\sum_{\tau=0}^{N-1}\sum_{i=1}^{m}\sum_{k=0}^{\infty}(\Sigma E_i\delta_\tau)^*(k)(\Sigma E_i\delta_\tau)(k)} \tag{2.5}$$

where $\delta_i = \delta(k-i)$ denotes the Dirac delta function and E_i is the ith column vector of the $m\times m$ identity matrix. This is a commonly used definition for periodic systems [1], and is an extension of the well known H_2 norm for linear time invariant systems.

We now apply the well known lifting technique [14, 18] to the periodic system Σ to obtain the following lifted system $\bar{\Sigma}_k$ for each $k \in [0, N-1]$.

$$(\bar{\Sigma}_k): \begin{aligned} \bar{x}^k_{l+1} &= F_k\bar{x}^k_l + G_k\bar{u}^k_l \qquad (2.6)\\ \bar{y}^k_l &= H_k\bar{x}^k_l + E_k\bar{u}^k_l \qquad (2.7)\end{aligned}$$

where

$$\bar{x}_l^k = x_{k+lN}, \quad \bar{u}_l^k = \begin{bmatrix} u_{k+lN} \\ u_{k+lN+1} \\ \vdots \\ u_{k+(l+1)N-1} \end{bmatrix}, \quad \bar{y}_l^k = \begin{bmatrix} y_{k+lN} \\ y_{k+lN+1} \\ \vdots \\ y_{k+(l+1)N-1} \end{bmatrix}, \tag{2.8}$$

$$F_k = \Phi_{k+N+1,k}, \tag{2.9}$$

$$G_k = \begin{bmatrix} \Phi_{k+N,k+1}B_k & \Phi_{k+N,k+2}B_{k+1} & \cdots & \Phi_{k+N,k+N-1}B_{k+N-2} & B_{k+N-1} \end{bmatrix}, \tag{2.10}$$

$$H_k = \begin{bmatrix} C_k \\ C_{k+1}\Phi_{k+1,k} \\ \vdots \\ C_{k+N-1}\Phi_{k+N-1,k} \end{bmatrix}, \tag{2.11}$$

$$E_k = \begin{bmatrix} D_k & 0 & 0 & \cdots & 0 & 0 \\ C_{k+1}B_k & D_{k+1} & 0 & \cdots & 0 & 0 \\ C_{k+2}\Phi_{k+2,k+1}B_k & C_{k+2}B_{k+1} & D_{k+2} & \cdots & 0 & 0 \\ \vdots & & & & & \\ C_{k+N-1}\Phi_{k+N-1,k+1}B_k & C_{k+N-1}\Phi_{k+N-1,k+2}B_{k+1} & & \cdots & C_{k+N-1}B_{k+N-2} & D_{k+N-1} \end{bmatrix} \tag{2.12}$$

The lifted system $\bar{\Sigma}_k$ is an equivalent expression of Σ in the linear time invariant state equation form (2.6)-(2.7). Let $\|\bar{\Sigma}_k\|_2$ be the H_2 norm of the linear time invariant system $\bar{\Sigma}_k$ following from the standard definition. It is straightforward to verify that

$$\|\bar{\Sigma}_i\|_2 = \|\bar{\Sigma}_j\|_2, \quad \forall\, i, j \in [0, N-1] \tag{2.13}$$

Using (2.5), it is also straightforward to verify

$$\|\Sigma\|_2 = \frac{1}{N}\|\bar{\Sigma}_k\|_2, \quad \forall k \in [0, N-1] \tag{2.14}$$

To formulate the optimal order reduction problem for periodic systems, we now give two useful technical results as follows.

LEMMA 2.1. *The N-periodic system Σ is stable if and only if for any N-periodic positive definite matrix $Q_k \in \mathcal{R}^{n_k\times n_k}$, with $Q_k = Q_k^T > 0$ and $Q_{k+N} = Q_k$, there exists a unique N-periodic positive definite solution $P_k \in \mathcal{R}^{n_k\times n_k}$, with $P_k = P_k^T > 0$ and $P_{k+N} = P_k$ for the following N-periodic Lyapunov equation*

$$A_k^T P_{k+1} A_k - P_k = -Q_k, \quad k \in [0, N-1] \tag{2.15}$$

Proof: Consecutively substituting $P_i = A_i^T P_{i+1} A_i + Q_i$ for $i = k+1,\ k+2,\ \cdots,\ k+N$ into (2.15) and using $P_{k+N} = P_k$, we obtain the following periodic Lyapunov equation.

$$\Phi_{k+N,k}^T P_k \Phi_{k+N,k} - P_k = -\bar{Q}_k, \quad k \in [0, N-1] \tag{2.16}$$

where

$$\bar{Q}_k = \bar{Q}_k^T = \sum_{i=k}^{k+N-1} \Phi_{i,k}^T Q_i \Phi_{i,k} > 0$$

Apparently, the Lyapunov equation (2.16) is an equivalent expression of (2.15).

By the well known Lyapunov stability result for linear time invariant systems and stability of the periodic system in terms of the transition matrices $\Phi_{k+N,k}$ for $k \in [0, N-1]$, the periodic system Σ is stable if and only if for the given Q_k and, consequently, $\bar{Q}_k$ there is a unique solution P_k for the Lyapunov equation (2.16). Thus the lemma is established following from that (2.15) is an equivalent expression of (2.16).

LEMMA 2.2. *The N-periodic system Σ is stable if and only if there exists a linear N-periodic transformation $z_k = T_k x_k$, i.e. $T_{k+N} = T_k$, such that the transformed N-periodic system*

$$z_{k+1} = T_{k+1} A_k T_k^{-1} z_k$$

satisfies $\|T_{k+1} A_k T_k^{-1}\| < 1$, $k \in [0, N-1]$.

Proof: If the periodic system Σ is stable, let the solution P_k for the Lyapunov equation (2.15) be written as $P_k = T_k^T T_k$, for $k \in [0, N-1]$. Pre-multiplying and post-multiplying (2.15) by T_k^{-T} and T_k^{-1}, respectively, yield

$$\left(T_{k+1} A_k T_k^{-1}\right)^T \left(T_{k+1} A_k T_k^{-1}\right) < I$$

It follows that

$$\|T_{k+1} A_k T_k^{-1}\| < 1$$

On the other hand, it is simple to show that if the system is unstable there exists no state transformation $z_k = T_k x_k$ such that $\|T_{k+1} A_k T_k^{-1}\| < 1$ for $k \in [0, N-1]$ is satisfied.

Throughout this paper, we assume that the periodic system Σ is stable. In view of Lemma 2.2, without loss of generality, we further assume that the state matrix of Σ satisfies $\|A_k\| < 1$ for $k \in [0, N-1]$.

2.2. H_2 OPTIMAL MODEL REDUCTION FOR PERIODIC SYSTEMS

We now state the H_2 optimal model reduction problem for periodic systems as: *For the given N-periodic system Σ with periodically time-varying order n_k and for a given desirable reduced periodically time-varying order $\hat{n}_k \leqslant n_k$ with $\sum_{k=0}^{N-1} \hat{n}_k < \sum_{k=0}^{N-1} n_k$, find an N-periodic model $\hat{\Sigma}$ with the reduced periodically time-varying order $\hat{n}_k$ of the following form such that $\|\Sigma - \hat{\Sigma}\|_2$ is minimized.*

$$(\hat{\Sigma}) : \hat{x}_{k+1} = \hat{A}_k \hat{x}_k + \hat{B}_k u_k \tag{2.17}$$
$$\hat{y}_k = \hat{C}_k \hat{x}_k + \hat{D}_k u_k \tag{2.18}$$

where $\hat{x}_k \in \mathcal{R}^{\hat{n}_k}$ is the state and and $\hat{A}_{k+N} = \hat{A}_k \in \mathcal{R}^{\hat{n}_{k+1} \times \hat{n}_k}$, $\hat{B}_{k+N} = \hat{B}_k \in \mathcal{R}^{\hat{n}_{k+1} \times m}$, $\hat{C}_{k+N} = \hat{C}_k \in \mathcal{R}^{p \times \hat{n}_k}$ and $\hat{D}_{k+N} = \hat{D}_k = D_k \in \mathcal{R}^{p \times m}$ are matrices of the reduced order system.

It is noted that the reduced order system has a periodically time-varying state dimension $\hat{n}_k$. The above problem statement takes into account the most general case to allow the state dimension to be reduced each time or only at some time and remain unchanged at some other time. The specific reduced state dimension is at the designer's choice.

Let the transition matrix of the reduced N-periodic system $\hat{\Sigma}$ be

$$\hat{\Phi}_{i,j} = \begin{cases} \hat{A}_{i-1} \hat{A}_{i-2} \cdots \hat{A}_j, & i > j \\ I, & i = j \end{cases}$$

The lifted system of the reduced order periodic system $\hat{\Sigma}_k$, for each $k \in [0, N-1]$, is obtained as

$$(\bar{\hat{\Sigma}}_k) : \bar{\hat{x}}_{l+1}^k = \hat{F}_k \bar{\hat{x}}_l^k + \hat{G}_k \bar{u}_l^k \tag{2.19}$$
$$\bar{\hat{y}}_l^k = \hat{H}_k \bar{\hat{x}}_l^k + \hat{E}_k \bar{u}_l^k \tag{2.20}$$

where

$$\bar{\hat{x}}_l^k = \hat{x}_{k+lN}, \quad \bar{u}_l^k = \begin{bmatrix} u_{k+lN} \\ u_{k+lN+1} \\ \vdots \\ u_{k+(l+1)N-1} \end{bmatrix}, \quad \bar{y}_l^k = \begin{bmatrix} y_{k+lN} \\ y_{k+lN+1} \\ \vdots \\ y_{k+(l+1)N-1} \end{bmatrix}, \tag{2.21}$$

$$\hat{F}_k = \hat{\Phi}_{k+N,k}, \tag{2.22}$$

(2.23)

$$\hat{G}_k = \left[\hat{\Phi}_{k+N,k+1} \hat{B}_k \quad \hat{\Phi}_{k+N,k+2} \hat{B}_{k+1} \quad \cdots \quad \hat{\Phi}_{k+N,k+N-1} \hat{B}_{k+N-2} \quad \hat{B}_{k+N-1} \right], \tag{2.24}$$

$$\hat{H}_k = \begin{bmatrix} \hat{C}_k \\ \hat{C}_{k+1}\hat{\Phi}_{k+1,k} \\ \vdots \\ \hat{C}_{k+N-1}\hat{\Phi}_{k+N-1,k} \end{bmatrix}, \tag{2.25}$$

$$\hat{E}_k =$$

$$\begin{bmatrix} D_k & 0 & 0 & \cdots & 0 & 0 \\ \hat{C}_{k+1}\hat{B}_k & D_{k+1} & 0 & \cdots & 0 & 0 \\ \hat{C}_{k+2}\hat{\Phi}_{k+2,k+1}\hat{B}_k & \hat{C}_{k+2}\hat{B}_{k+1} & D_{k+2} & \cdots & 0 & 0 \\ \vdots & & & & & \\ \hat{C}_{k+N-1}\hat{\Phi}_{k+N-1,k+1}\hat{B}_k & \hat{C}_{k+N-1}\hat{\Phi}_{k+N-1,k+2}\hat{B}_{k+1} & & \cdots & \hat{C}_{k+N-1}\hat{B}_{k+N-2} & D_{k+N-1} \end{bmatrix} \tag{2.26}$$

The model reduction problem for linear time-invariant systems has been studied under various performance measures. Commonly used methods for linear time invariant system model reduction include the balanced truncation [20] and the Hankel-norm approximation [7]. These methods are error-bound based but are not optimal in the sense of minimizing certain error specification between the original model and the reduced order model.

In [12], it has been shown that for an nth order linear time-invariant model (A, B, C), the optimal rth order H_2 reduced model must be of form (UAV, UB, CV) where $U \in \mathcal{R}^{r\times n}$ and $V \in \mathcal{R}^{n\times r}$ must satisfy the constraints that UAV is stable and $UV = I$. These nonlinear constraints impose a formidable difficulty for efficiently computing the matrix U and V for the optimal model reduction.

To deal with this difficulty, a modified H_2 model reduction problem for continuous time systems is considered in [29, 30] where the reduced model is of the form $(U^T AU, UB, CU)$ with the constraint $U^T U = I$. There are a number of valid reasons for considering the H_2 model reduction of this form. Firstly, it includes the balanced truncation as a special case and provides a reasonably good approximation to the original model reduction problem; (see [30]). Next, the minimal solution for the model reduction exists since the set $\{U^T U = I\}$ is compact. Further, the stability constraint for the reduced order model is removed. Rigorous convergent algorithms are given in [30] and illustrated by examples.

For linear periodic system model reduction, a Hankel-norm approximation approach is proposed in [28] where the periodic systems are required to be reversible and have a constant order, i.e. $n_k = n$, for all $k \geqslant 0$.

In this paper, we consider that the reduced order system $\hat{\Sigma}$ is of the form

$$\hat{A}_k = U_k^T A_k V_k, \quad \hat{B}_k = U_k^T B_k, \quad \hat{C}_k = C_k V_k \tag{2.27}$$

where U_k and V_k are N-periodic real matrices from the Stiefel manifolds:

$$S_{uk} = \left\{ U_k \in \mathcal{R}^{n_{k+1}\times \hat{n}_{k+1}} \mid U_k^T U_k = I \right\} \tag{2.28}$$

$$S_{vk} = \left\{ V_k \in \mathcal{R}^{n_k\times \hat{n}_k} \mid V_k^T V_k = I \right\} \tag{2.29}$$

Observe that the set of reduced order models characterized in (2.26) is larger than that of [30] where only one projection operator is used. Also, the reduced order periodic system (2.26) is always stable for any $U_k \in S_{uk}$ and $V_k \in S_{vk}$ under the assumption $\|A_k\| < 1$. Therefore, there is no stability constraint in computing the solution for U_k and V_k.

In view of the H_2 norm properties (2.13) and (2.14) of the lifted systems, the H_2 optimal model reduction problem for linear periodic systems can be further formulated in terms of the lifted systems as: *find matrices $U_k \in S_{uk}$ and $V_k \in S_{vk}$, $k \in [0, N-1]$ for the reduced order periodic system matrices in (2.26) such that $\|\bar{\Sigma}_0 - \bar{\hat{\Sigma}}_0\|_2$ is minimized.*

3. The H_2 cost function and its gradient

The error model $\bar{\Sigma}_0 - \bar{\hat{\Sigma}}_0$ is a linear time invariant system, which has a state-space realization as follows.

$$\zeta_{l+1} = F_c \zeta_l + G_c u_l^0 \tag{3.1}$$
$$e_l = H_c \zeta_l + E_c u_l^0 \tag{3.2}$$

where $\zeta_l = \begin{bmatrix} \bar{x}_l^{0T} & \bar{\hat{x}}_l^{0T} \end{bmatrix}^T$ and

$$F_c = \begin{bmatrix} F_0 & 0 \\ 0 & \hat{F}_0 \end{bmatrix}, \tag{3.3}$$
$$G_c = \begin{bmatrix} G_0 \\ \hat{G}_0 \end{bmatrix}, \tag{3.4}$$
$$H_c = [H_0 \quad -\hat{H}_0], \tag{3.5}$$
$$E_c = E_0 - \hat{E}_0 \tag{3.6}$$

where $F_0,\ G_0,\ H_0,\ E_0,\ \hat{F}_0,\ \hat{G}_0,\ \hat{H}_0$ and $\hat{E}_0$ are as given in (2.9)-(2.12) and (2.22)-(2.25). It is well known [19] that

$$\mathcal{J}(U, V) = \|\bar{\Sigma}_0 - \bar{\hat{\Sigma}}_0(U, V)\|_2^2 = \mathcal{J}_1(U, V) + \mathcal{J}_2(U, V) \tag{3.7}$$

where

$$U = (U_0, U_1, \cdots, U_{N-1}) \tag{3.8}$$
$$V = (V_0, V_1, \cdots, V_{N-1}) \tag{3.9}$$
$$\mathcal{J}_1(U, V) = \mathrm{tr}(H_c P H_c^T) = \mathrm{tr}(G_c^T Q G_c) \tag{3.10}$$
$$\mathcal{J}_2(U, V) = \mathrm{tr}\{(E_0 - \hat{E}_0)^T (E_0 - \hat{E}_0)\} \tag{3.11}$$

and P and Q satisfy the following Lyapunov equations, respectively.

$$F_c P F_c^T - P + G_c G_c^T = 0 \tag{3.12}$$
$$F_c^T Q F_c - Q + H_c^T H_c = 0 \tag{3.13}$$

Note that the matrices P, Q, F_c, G_c, H_c and E_c, $\mathcal{J}_1(U, V)$ and $\mathcal{J}_2(U, V)$ are differentiable functions of (U, V) and $\mathcal{J}(U, V)$ is a smooth function on the manifold

$$\Psi = S_{u0} \times S_{u1} \times \cdots \times S_{u(N-1)} \times S_{v0} \times S_{v1} \times \cdots \times S_{v(N-1)}$$

It can be shown ([9]) that the tangent space of Ψ at any $(U, V) \in \Psi$ is given by

$$T_{(U,V)}\Psi = \left\{(\xi, \eta) \mid \xi_j^T U_j + U_j^T \xi_j = 0,\ \eta_j^T V_k + V_j^T \eta_j = 0,\ j = 0, 1, \cdots, N-1\right\} \tag{3.14}$$

where $\xi_j \in \mathcal{R}^{n_{j+1}\times \hat{n}_{j+1}}$, $\eta_j \in \mathcal{R}^{n_j \times \hat{n}_j}$, $\xi = (\xi_1, \xi_2, \cdots, \xi_{N-1})$ and $\eta = (\eta_1, \eta_2, \cdots, \eta_{N-1})$.

By endowing $T_{(U,V)}\Psi$ with the inner product

$$< (\xi, \eta), (\xi', \eta') >= \sum_{j=0}^{N-1} \operatorname{tr}\left(\xi_j^T \xi_j' + \eta_j^T \eta_j'\right),\ \forall(\xi, \eta), (\xi', \eta') \in T_{(U,V)}\Psi, \tag{3.15}$$

Ψ becomes a Riemannian manifold.

Let the gradient of the H_2 cost function $\mathcal{J}(U, V)$ on the tangent space $T_{(U,V)}\Psi$ be

$$\nabla\mathcal{J}(U, V) = [\nabla\mathcal{J}_U(U, V)\ \ \nabla\mathcal{J}_V(U, V)] \tag{3.16}$$

where

$$\nabla\mathcal{J}_U(U, V) = \left[\nabla\mathcal{J}_{U_0}(U, V)\ \ \nabla\mathcal{J}_{U_1}(U, V)\ \cdots\ \nabla\mathcal{J}_{U_{N-1}}(U, V)\right] \tag{3.17}$$

$$\nabla\mathcal{J}_V(U, V) = \left[\nabla\mathcal{J}_{V_0}(U, V)\ \ \nabla\mathcal{J}_{V_1}(U, V)\ \cdots\ \nabla\mathcal{J}_{V_{N-1}}(U, V)\right] \tag{3.18}$$

In view of (3.4), (3.5), (2.10), (2.11) (2.23) and (2.24), we partition G_c and H_c as

$$G_c = \left[G_{c0}\ \ G_{c1}\ \cdots\ G_{c(N-1)}\right] \tag{3.19}$$

$$H_c = \left[H_{c0}^T\ \ H_{c1}^T\ \cdots\ H_{c(N-1)}^T\right]^T \tag{3.20}$$

For the notational simplicity, let $\hat{\Phi}_{i,j} = 0$ when $i < j$ and introduce

$$R_{1j} = A_j V_j P_j F_c^T Q_j^T,\quad S_{1j} = A_j^T U_j Q_j F_c P_j^T \tag{3.21}$$

$$R_{2j} = \left(B_j G_{cj}^T + \sum_{k=1}^{j} A_j V_j \hat{\Phi}_{j,k} \hat{B}_{k-1} G_{c(k-1)}^T\right) Q_j^T \tag{3.22}$$

$$S_{2j} = \sum_{k=1}^{j} A_j^T U_j Q_j G_{c(k-1)} \hat{B}_{k-1}^T \hat{\Phi}_{j,k}^T \tag{3.23}$$

$$R_{3j} = -\sum_{k=j+1}^{N-1} A_j V_j P_j H_{ck}^T \hat{C}_k \hat{\Phi}_{k,j+1} \tag{3.24}$$

$$S_{3j} = -\left(C_j^T H_{cj} + \sum_{k=j+1}^{N-1} A_j^T U_j \hat{\Phi}_{k,j+1}^T \hat{C}_k^T H_{ck} \right) P_j^T \tag{3.25}$$

$$R_{4j} = \sum_{k=1}^{N-1} \sum_{l=j+1}^{N-1} A_j V_j \hat{\Phi}_{j,k} \hat{B}_{k-1} \Gamma_{lk}^T \hat{C}_l \hat{\Phi}_{l,j+1} + \sum_{k=j+1}^{N-1} B_j \Gamma_{j(k+1)}^T \hat{C}_k \hat{\Phi}_{k,j+1} \tag{3.26}$$

$$S_{4j} = \sum_{k=1}^{N-1} \sum_{l=j+1}^{N-1} A_j^T U_j \hat{\Phi}_{l,j+1}^T \hat{C}_l^T \Gamma_{lk} \hat{B}_{k-1}^T \hat{\Phi}_{j,k}^T + \sum_{k=1}^{j} C_j^T \Gamma_{jk} \hat{B}_{k-1}^T \hat{\Phi}_{j,k}^T \tag{3.27}$$

$$R_j = R_{1j} + R_{2j} + R_{3j} + R_{4j}, \quad S_j = S_{1j} + S_{2j} + S_{3j} + S_{4j} \tag{3.28}$$

where

$$P_j = \hat{\Phi}_{j,0}[0 \;\; I]P, \quad Q_j = \hat{\Phi}_{N,j+1}^T[0 \;\; I]Q, \quad \Gamma_{jk} = \hat{C}_j \hat{\Phi}_{j,k} \hat{B}_{k-1} - C_j \Phi_{j,k} B_{k-1} \tag{3.29}$$

Using these notations, we present a result on the gradient $\nabla \mathcal{J}(U, V)$ in the following theorem.

THEOREM 1. *The gradient of the H_2 cost function $\mathcal{J}(U, V)$ on the tangent space $T_{(U,V)}\Psi$ satisfies*

$$\nabla \mathcal{J}_{U_j}(U, V) = 2R_j - U_j(R_j^T U_j + U_j^T R_j), \quad \nabla \mathcal{J}_{V_j}(U, V) = 2S_j - V_j(S_j^T V_j + V_j^T S_j), \quad j \in [0, N-1] \tag{3.30}$$

Proof: See Appendix.

It is noted that the gradient on the tangent space $\nabla \mathcal{J}(U, V)$ is different from the gradient of $\mathcal{J}(U, V)$ as a usual function defined on $(U, V) \in \Psi$.

4. Algorithms for minimizing the H$_2$ cost function

It is known from calculus that a first order condition for minimality of the H_2 cost function $\nabla \mathcal{J}(U, V)$ is

$$\nabla \mathcal{J}_{U_j}(U, V) = 0, \quad \nabla \mathcal{J}_{V_j}(U, V) = 0, \quad (U, V) \in \Psi, \quad j \in [0, N-1]$$

Applying Theorem 1 leads to

$$2R_j - U_j(R_j^T U_j + U_j^T R_j) = 0, \quad 2S_j - V_j(S_j^T V_j + V_j^T S_j) = 0, \quad (U, V) \in \Psi, \quad j \in [0, N-1] \tag{4.1}$$

In general, the solutions U_j and V_j, $j \in [0, N-1]$, for these equations are not straightforward and there is no guarantee that a solution for these equations yields a minimal cost function. To deal with this problem, we use a gradient flow approach to develop two algorithms in this section for computing U and V to obtain an optimal reduced order periodic model. The first algorithm is based on solving differential matrix equations in continuous time and the second is a discrete recursive algorithm which is simple as far as implementation is concerned. Convergence properties will be established for the algorithms.

4.1. THE CONTINUOUS TIME ALGORITHM

It is known that the value of the cost function is decreased along the negative gradient direction. Thus, we form a gradient flow as follows to approach a minimal solution $(U, V) \in \Psi$:

$$\dot{U}_j = U_j(R_j^T U_j + U_j^T R_j) - 2R_j \tag{4.2}$$
$$\dot{V}_j = V_j(S_j^T V_j + V_j^T S_j) - 2S_j \tag{4.3}$$

Since the optimization of the cost $\mathcal{J}(U, V)$ over the manifold Ψ is a constrained optimization problem, we need to ensure that the solution (U, V) for (4.2) and (4.3) exists and remains in Ψ for an initial condition $(U(0), V(0)) \in \Psi$. We give useful properties of the gradient flow in the following theorem, which will lead to the implementation of the gradient-based algorithm.

THEOREM 2. *For an initial condition $(U(0), V(0)) \in \Psi$, the solution for the ordinary differential equations (ODEs) (4.2) and (4.3) satisfy*

1. *The solution $(U(t), V(t))$ for (4.2) and (4.3) is unique and stays in Ψ for all $t \geqslant 0$;*
2. *The cost $\mathcal{J}(U, V)$ is strictly decreasing to approach the minimum solution in the sense*

$$\mathcal{J}(U(t_2), V(t_2)) - \mathcal{J}(U(t_1), V(t_1)) = -\sum_{j=0}^{N-1} \int_{t_1}^{t_2} M_j(U, V)dt, \quad \forall t_2 \geqslant t_1 \geqslant 0 \tag{4.4}$$

   where

$$M_j(U, V) = \sum_{j=0}^{N-1} \left\{ \|U_j R_j^T - R_j U_j^T\|_F^2 + 2\|(U_j U_j^T - I)R_j\|_F^2 + \|V_j S_j^T - S_j V_j^T\|_F^2 + 2\|(V_j V_j^T - I)S_j\|_F^2 \right\} \geqslant 0 \tag{4.5}$$

   and $\|\cdot\|_F$ denotes the Frobenius norm;
3. *The derivatives $\dot{U}(t)$ and $\dot{V}(t)$ converge to zero, i.e.*

$$\lim_{t\to\infty} \dot{U}_j(t) = 0, \quad \lim_{t\to\infty} \dot{V}_j(t) = 0, \quad j \in [0, N-1]$$

4. *Any solution* $(U^*(t), V^*(t))$ *for (4.2) and (4.3), such that the cost function* $\mathcal{J}(U^*(t), V^*(t))$ *is minimal, satisfies*

$$M_j(U^*, V^*) = 0$$

or equivalently,

$$U_j^* R_j^T - R_j U_j^{*T} = 0, \quad V_j^* S_j^T - S_j V_j^{*T} = 0, \quad j \in [0, N-1]$$

Proof: 1) Since Ψ is a compact set, the solution for the ODEs is unique. Using (4.2) and (4.3), it can be easily shown that

$$\frac{d}{dt}(U_j^T(t) U_j(t)) = 0 \quad \text{and} \quad \frac{d}{dt}(V_j^T(t) V_j(t)) = 0$$

Thus, $(U(t), V(t)) \in \Psi$ for all $t \geqslant 0$ provided $(U(0), V(0)) \in \Psi$.

2) The result (4.4) follows immediately from

$$\begin{aligned}
\dot{\mathcal{J}}[U(t), V(t)] &= 2\sum_{j=0}^{N-1} \mathrm{tr}\left(R_j^T \dot{U}_j + S_j^T \dot{V}_j\right) \\
&= 2\sum_{j=0}^{N-1} \mathrm{tr}\left\{R_j^T U_j R_j^T U_j - R_j^T R_j + R_j^T (U_j U_j^T - I) R_j\right\} \\
&\quad +2\sum_{j=0}^{N-1} \mathrm{tr}\left\{S_j^T V_j S_j^T V_j - S_j^T S_j + S_j^T (V_j V_j^T - I) S_j\right\} \\
&= -\sum_{j=0}^{N-1} \mathrm{tr}\left\{(U_j R_j^T - R_j U_j^T)^T (U_j R_j^T - R_j U_j^T)\right\} \\
&\quad + 2\sum_{j=0}^{N-1} \mathrm{tr}\left\{R_j^T (U_j U_j^T - I) R_j\right\} \\
&\quad -\sum_{j=0}^{N-1} \mathrm{tr}\left\{(V_j S_j^T - S_j V_j^T)^T (V_j S_j^T - S_j V_j^T)\right\} \\
&\quad + 2\sum_{j=0}^{N-1} \mathrm{tr}\left\{S_j^T (V_j V_j^T - I) S_j\right\} \\
&= -\sum_{j=0}^{N-1} M_j(U, V)
\end{aligned}$$

3) If $M(U^*, V^*) \neq 0$, since $0 \leqslant \mathcal{J}(U(t), V(t)) \leqslant \mathcal{J}(U(0), V(0))$, the integral

$$\int_0^\infty \|M_j(U, V)\|_F^2 dt$$

must be finite. By the uniform continuity of $(\dot{U}(t), \dot{V}(t))$ for any $t \geqslant 0$, we have

$$\lim_{t\to\infty}(U_j R_j^T - R_j U_j^T) = 0, \quad \lim_{t\to\infty}(U_j U_j^T - I)R_j = 0, \quad j \in [0, N-1] \tag{4.6}$$

$$\lim_{t\to\infty}(V_j S_j^T - S_j V_j^T) = 0, \quad \lim_{t\to\infty}(V_j V_j^T - I)S_j = 0 \quad j \in [0, N-1] \tag{4.7}$$

These yield

$$R_j = U_j R_j^T U_j = U_j U_j^T R_j, \quad S_j = V_j S_j^T V_j = V_j V_j^T S_j$$

It then follows from (4.2) and (4.3) that $(\dot{U}_j(t), \dot{V}_j(t)) \to 0$.

4) If $(U^*(t), V^*(t))$ is a solution such that the cost function $\mathcal{J}(U^*(t), V^*(t))$ is minimal, there must be $\lim_{t\to\infty} U(t) = U^*$ and $\lim_{t\to\infty} V(t) = V^*$. Then the result follows from that of 3).

Theorem 2 shows that, starting from an initial condition in Ψ, a critical solution of (4.2) and (4.3) can be obtained by integrating the differential equations using any numerical packages such as Matlab. Theorem 2 guarantees that the cost function $\mathcal{J}(U, V)$ decreases monotonically and converges to a minimal value. It is also shown that if the cost function has only isolated minimum points, the solution $(U(t), V(t))$ will converge to one of them.

For a given stable N-periodic system Σ satisfying $\|A_k\| < 1$ for $k \geqslant 0$, we summarize the gradient flow algorithm for the H_2 model reduction based on Theorem 2 as follows.

1. Obtain the lifted system of Σ in the form (2.19)–(2.20);
2. Choose $(U(0), V(0)) \in \Psi$ and specify a time duration t_f for integration;
3. Obtain R_j and S_j for $j \in [0, N-1]$ as given in (3.28), which are functions of U and V;
4. Integrate the ODEs (4.2) and (4.3);
5. If the final cost $\mathcal{J}(U(t_f), V(t_f))$ is not satisfactory, set $(U(0), V(0)) = (U(t_f), V(t_f))$ and repeat Step 3.

4.2. THE DISCRETE ITERATIVE ALGORITHM

This subsection gives a discrete iterative algorithm for solving the ODEs (4.2) and (4.3). This is to meet the need of digital computation and implementation for the model reduction. Let

$$\Xi_j(k) = U_j(k)R_j^T - R_j U_j(k)^T, \quad \Pi_j(k) = V_j(k)S_j^T - S_j V_j(k)^T \tag{4.8}$$

and define the following iterations:

$$U_j(k+1) = e^{t_k \Xi_j(k)} U_j(k), \quad V_j(k+1) = e^{t_k \Pi_j(k)} V_j(k), \quad j = 0, 1 \cdots, N-1 \tag{4.9}$$

where t_k is a step-size to be chosen. We show in the following theorem that (4.9) can be used to iteratively compute the gradient flow (4.2) and (4.3).

THEOREM 3. *There exists a constant c such that for a step-size t_k with $0 < t_k < c$ and for an initial condition $(U(0), V(0)) \in \Psi$, $U(k)$ and $V(k)$ recursively computed from the equations (4.9) satisfy the following properties.*

1. *$(U(k), V(k)) \in \Psi$ for any $k > 0$.*
2. *The value of $\mathcal{J}(U(k), V(k))$ decreases monotonically as k increase, i.e.*

$$\mathcal{J}(U(k+1), V(k+1)) \leqslant \mathcal{J}(U(k), V(k)), \quad \forall k \geqslant 0$$

where the equality holds if and only if a critical point is reached, i.e. $M_j(U(k), V(k)) = 0$ with $M_j(\cdot, \cdot)$ as in (4.5).

3.

$$\lim_{k\to\infty} M_j(U(k), V(k)) = 0, \quad j \in [0, N-1]$$

or equivalently

$$\lim_{k\to\infty} \left[U_j(k)R_j^T - R_j U_j(k)^T\right] = 0, \quad \lim_{k\to\infty} \left[V(k)S_j^T - S_j V(k)^T\right] = 0, \quad j \in [0, N-1]$$

Proof: 1) Observed from (4.8) that $\Phi_j(k)$ and $\Pi_j(k)$ are skew-symmetric. Thus the matrices $e^{t_k \Phi_j(k)}$ and $e^{t_k \Pi_j(k)}$ are orthogonal for any t_k. It follows that $(U(k), V(k)) \in \Psi$ if $(U(0), V(0)) \in \Psi$.

2) Let

$$U_j(t) = e^{t\Xi_j(k)} U_j(k), \quad V_j(t) = e^{t\Pi_j(k)} V_j(k)$$

and $R_j(t)$ and $S_j(t)$ be the corresponding R_j and S_j as in (3.28). Clearly, $(U(0), V(0)) = (U(k), V(k))$, $R_j(0) = R_j(k)$ and $S_j(0) = S_j(k)$. By the Taylor expansion, there exists a $\theta \in [0, t]$ such that

$$\mathcal{J}(U(t), V(t)) - \mathcal{J}(U(k), V(k)) = t\dot{\mathcal{J}}(U(0), V(0)) + \frac{t^2}{2}\ddot{\mathcal{J}}(U(\theta), V(\theta)) \tag{4.10}$$

It can be seen from (A.21) that

$$\begin{aligned} \dot{\mathcal{J}}(U(t), V(t)) &= 2\sum_{j=0}^{N-1} \mathrm{tr}\left(R_j^T \dot{U}_j + S_j^T \dot{V}_j\right) \\ &= 2\sum_{j=0}^{N-1} \mathrm{tr}\left(R_j^T \Xi_j(k) U_j + S_j^T \Pi_j(k) V_j\right) \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} \ddot{\mathcal{J}}(U(t), V(t)) = 2\sum_{j=0}^{N-1} \mathrm{tr}\big(&\dot{R}_j^T \Xi_j(k) U_j + R_j^T \Xi_j(k)^2 U_j \\ &+ \dot{S}_j^T \Pi_j(k) V_j + S_j \Pi_j(k)^2 V_j\big) \end{aligned} \tag{4.12}$$

By observing that $\Xi_j(k)^T = -\Xi_j(k)$, $\Pi_j(k) = -\Pi_j(k)^T$ and using (4.9), it can be shown that

$$\begin{aligned}\dot{\mathcal{J}}(U(0), V(0)) &= -\sum_{j=0}^{N-1}\left\{\operatorname{tr}(\Xi_j(k)^T\Xi_j(k)) + \operatorname{tr}(\Pi_j(k)^T\Pi_j(k))\right\}\\ &= -\sum_{j=0}^{N-1}(\|\Xi_j(k)\|_F^2 + \|\Pi_j(k)\|_F^2)\end{aligned} \tag{4.13}$$

Further, we have

$$\begin{aligned}|\ddot{\mathcal{J}}(U(t), V(t))| \leqslant 2\,(\|\dot{R}_j\|_F\|\Xi_j(k)\|_F + \|R_j\|_F\|\Xi_j(k)^2\|_F\\ +\|\dot{S}_j\|_F\|\Pi_j(k)\|_F + \|S_j\|_F\|\Pi_j(k)^2\|_F\big)\end{aligned} \tag{4.14}$$

Similar to the proof of Theorem 4.1 in [29], it can be shown that there exist positive constants $\alpha_j, \beta_j, \gamma_j$ and τ_j such that, for $j \in [0, N-1]$,

$$\begin{aligned}\|R_j\|_F \leqslant \alpha_j, \quad \|S_j\|_F \leqslant \gamma_j, \quad \|\dot{R}_j\|_F \leqslant \beta_j\||\Phi_j(k)\|_F,\\ \|\dot{S}_j\|_F \leqslant \tau_j\||\Pi_j(k)\|_F\end{aligned} \tag{4.15}$$

Therefore, it then follows that

$$\begin{aligned}|\ddot{\mathcal{J}}(U(t), V(t))| \leqslant 2\sum_{j=0}^{N-1}\big(\alpha_j\|\Xi_j(k)\|_F^2 + \beta_j\|\Xi_j(k)\|_F^2 + \gamma_j\|\Pi_j(k)\|_F^2\\ +\tau_j\|\Pi_j(k)\|_F^2\big)\end{aligned} \tag{4.16}$$

Hence, it is obtained from (4.10), (4.13) and (4.16) that

$$\begin{aligned}&\mathcal{J}(U(t), V(t)) - J(U_k, V_k)\\ &\leqslant \sum_{j=0}^{N-1}\left\{\left[-t + \left(\alpha_j + \beta_j\right)t^2\right]\|\Xi_j(k)\|_F^2 + \left[-t + \left(\gamma_j + \tau_j\right)t^2\right]\|\Pi_j(k)\|_F^2\right\}\end{aligned} \tag{4.17}$$

Let

$$c = \min\left\{\frac{1}{\alpha_j + \beta_j}, \frac{1}{\gamma_j + \tau_j},\ j = 0, 1, \cdots, N-1\right\}$$

It is obvious that when $t \in (0, s)$, $\mathcal{J}(U(t), V(t)) \leqslant \mathcal{J}(U(k), V(k))$, i.e. $\mathcal{J}(U(k+1), V(k+1)) \leqslant \mathcal{J}(U(k), V(k))$.

3) Take a step-size $t_k \in (0, c)$. It follows from (4.17) that

$$\sum_{j=0}^{N-1}\left(\|\Xi_j(k)\|_F^2 + \|\Pi_j(k)\|_F^2\right) \leqslant \frac{\mathcal{J}(U(k), V(k)) - \mathcal{J}(U(k+1), V(k+1))}{d} \tag{4.18}$$

where

$$d = t_k - \frac{1}{c} t_k^2 > 0$$

Taking the limit on the both sides of (4.18), we have

$$\lim_{k\to\infty} \Xi_j(k) = \lim_{k\to\infty} \Pi_j(k) = 0$$

Note that

$$\begin{aligned}
&(I - U_j(k)U_j(k)^T)R_j(k) \\
&= \left[I - U_j(k)U_j(k)^T\right]\left[R_j(k)U_j(k)^T - U_j(k)R_j(k)^T\right]U_j(k) \\
&(I - V_j(k)V_j(k)^T)S_j(k) \\
&= \left[I - V_j(k)V_j(k)^T\right]\left[S_j(k)V_j(k)^T - V_j(k)S_j(k)^T\right]V_j(k)
\end{aligned}$$

Then it is follows from (4.5) that $M_j(U(k), V(k)) = 0$ if and only if $\Gamma_k = 0$ and $\Pi_k = 0$. This completes the proof.

By appropriately choosing a constant step-size $t_k < c$, Theorem 3 guarantees that the H_2 cost function $\mathcal{J}(U(k), V(k))$ obtained from iterative computation of $U(k)$ and $V(k)$ by (4.9) converges to a minimal value. A procedure for the iterative computation is given as follows.

1. Obtain the lifted system of Σ in the form (2.19)-(2.20);
2. Choose $(U(0), V(0)) \in \Psi$ and choose an appropriate step-size t_k;
3. Obtain R_j and S_j for $j \in [0, N-1]$ as given in (3.28), which are functions of U and V;
4. Compute $(U(k+1), V(k+1))$ from $(U(k), V(k))$ by using (4.9) and (4.8) until $\|\Xi_j(k)\| < \varepsilon$ and $\|\Pi_j(k)\| < \varepsilon$, $j \in [0, N-1]$, are satisfied for a specified tolerance $\varepsilon > 0$.

5. Examples

In this section we apply the discrete iterative algorithm for periodic system model reduction to two examples. The first example is for model reduction of a periodic system with a constant order and the second example is for a periodic system with a time varying order. The results obtained from the continuous time algorithm are similar and omitted.

Example 1

Consider a fourth order two-periodic system in the form (2.1)-(2.2) with

$$A_0 = \begin{bmatrix} 0.953 & 0.140 & 0.000 & 0.000 \\ -0.140 & 0.953 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.447 & 0.714 \\ 0.000 & 0.000 & -0.714 & 0.447 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0.300 & 0.000 \\ 0.500 & -0.300 \\ 0.150 & 0.000 \\ 0.000 & 0.200 \end{bmatrix},$$

$$C_0 = \begin{bmatrix} 0.500 & -0.400 & 0.500 & 0.300 \\ 0.000 & 0.000 & 0.400 & 0.300 \end{bmatrix}, \quad D_0 = \begin{bmatrix} 0.500 & 0.000 \\ 0.000 & 0.500 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} -0.600 & 0.100 & 0.000 & 0.300 \\ -0.300 & -0.700 & 0.000 & 0.000 \\ 0.580 & 0.000 & 0.400 & 0.300 \\ 0.100 & 0.000 & -0.600 & -0.400 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.200 & 0.000 \\ 0.000 & 0.200 \\ -0.300 & 0.500 \\ 0.000 & 0.400 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0.400 & -0.200 & 0.000 & 0.000 \\ 0.100 & 0.200 & 0.000 & 0.500 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0.600 & 0.000 \\ 0.000 & 0.600 \end{bmatrix}$$

It can be easily shown that this system satisfies $\|A_0\| < 1$ and $\|A_1\| < 1$. Therefore, we can apply the iterative algorithm directly. Our aim is to find a second order periodic model to approximate the above periodic system. Take the following initial condition

$$U_0 = V_0 = U_1 = V_1 = \begin{bmatrix} I_{2\times 2} \\ 0_{2\times 2} \end{bmatrix}$$

and choose the step-size to be $t_k = 0.01$. The cost evolution of the iterative algorithm is shown in Figure 1, which shows that the cost function decreases monotonically from 0.634 to 0.317. The projection matrices U_0, V_0, U_1 and V_1 converge to

$$U_0 = \begin{bmatrix} 0.9946 & -0.0037 \\ -0.0035 & 0.9646 \\ -0.0745 & 0.1280 \\ 0.0717 & 0.2307 \end{bmatrix}, \quad V_0 = \begin{bmatrix} 0.3527 & 0.0963 \\ 0.3366 & 0.9087 \\ 0.6544 & -0.3685 \\ 0.5780 & -0.1707 \end{bmatrix},$$

$$U_1 = \begin{bmatrix} 0.6643 & 0.1119 \\ 0.2103 & 0.8681 \\ 0.5108 & -0.4832 \\ 0.5036 & -0.0200 \end{bmatrix}, \quad V_1 = \begin{bmatrix} 0.9915 & -0.0101 \\ 0.0182 & 0.9762 \\ 0.1090 & 0.0602 \\ -0.0687 & 0.2083 \end{bmatrix}$$

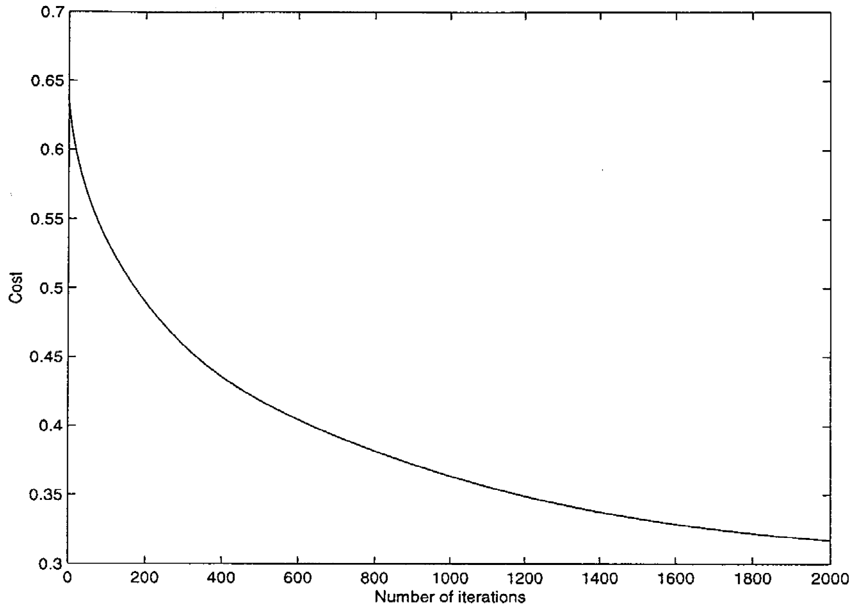

Figure 1. The cost evolution of Example 1.

As a result, the reduced order periodic model in the form (2.17)-(2.18) is obtained as

$$
\begin{aligned}
\hat{A}_0 &= \begin{bmatrix} 0.3127 & 0.2496 \\ 0.3025 & 0.8279 \end{bmatrix}, \quad \hat{B}_0 = \begin{bmatrix} 0.2855 & 0.0154 \\ 0.5004 & -0.2432 \end{bmatrix}, \\
\hat{C}_0 &= \begin{bmatrix} 0.5423 & -0.5508 \\ 0.4352 & -0.1986 \end{bmatrix}, \quad D_0 = \begin{bmatrix} 0.5000 & 0.0000 \\ 0.0000 & 0.5000 \end{bmatrix}, \\
\hat{A}_1 &= \begin{bmatrix} -0.1366 & -0.0521 \\ -0.6281 & -0.6086 \end{bmatrix}, \quad \hat{B}_1 = \begin{bmatrix} -0.0204 & 0.4989 \\ 0.1673 & -0.0760 \end{bmatrix}, \\
\hat{C}_1 &= \begin{bmatrix} 0.3930 & -0.1993 \\ 0.0685 & 0.2984 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0.6000 & 0.000 \\ 0.000 & 0.6000 \end{bmatrix}
\end{aligned}
$$

Example 2
Consider a two-periodic system with time-varying order $n_0 = 4$ and $n_1 = 5$ in

the form (2.1)–(2.2) with

$$A_0 = \begin{bmatrix} 0.883 & 0.140 & 0.000 & 0.000 \\ -0.140 & 0.883 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.447 & 0.714 \\ 0.000 & 0.000 & -0.714 & 0.447 \\ 0.100 & 0.200 & 0.000 & 0.100 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 1.000 & 0.000 \\ 0.500 & -0.300 \\ 0.150 & 0.000 \\ 0.000 & 2.000 \\ 0.000 & 0.200 \end{bmatrix},$$

$$C_0 = \begin{bmatrix} 0.500 & -0.400 & 1.000 & 0.300 \\ 0.000 & 0.000 & 0.400 & 0.300 \end{bmatrix}, \quad D_0 = \begin{bmatrix} 0.500 & 0.000 \\ 0.000 & 0.500 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} -0.500 & 0.100 & 0.000 & 0.300 & 0.100 \\ -0.200 & 0.500 & 0.000 & 0.000 & 0.000 \\ 0.300 & 0.000 & 0.350 & 0.650 & 0.100 \\ 0.100 & 0.000 & -0.500 & -0.300 & 0.000 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1.500 & 0.000 \\ 0.000 & 0.200 \\ -0.300 & 0.500 \\ 0.000 & 0.400 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 0.400 & -0.200 & 0.000 & 0.000 & 0.000 \\ 1.000 & 0.200 & 0.000 & 0.500 & 0.000 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0.600 & 0.000 \\ 0.000 & 0.600 \end{bmatrix}$$

It is easy to check that $\|A_0\| < 1$ and $\|A_1\| < 1$. To find a reduced second order model, we set the initial condition to be

$$U_0 = V_1 = \begin{bmatrix} I_{2\times 2} \\ 0_{3\times 2} \end{bmatrix}, \quad U_1 = V_0 = \begin{bmatrix} I_{2\times 2} \\ 0_{2\times 2} \end{bmatrix},$$

and the step-size to be $t_k = 0.005$. Applying the iterative algorithm yields the cost function evolution as shown Figure 2. It is shown that the cost function is reduced monotonically from 2.4197 to 0.6477. The projection matrices converge to

$$U_0 = \begin{bmatrix} 0.8849 & 0.3960 \\ 0.0594 & 0.1814 \\ 0.0258 & 0.3217 \\ 0.4431 & -0.8404 \\ 0.1281 & 0.0226 \end{bmatrix}, \quad V_0 = \begin{bmatrix} 0.9630 & -0.0242 \\ 0.2582 & 0.2039 \\ 0.0313 & -0.9785 \\ -0.0708 & -0.0176 \end{bmatrix},$$

$$U_1 = \begin{bmatrix} 0.9736 & -0.0510 \\ 0.1939 & 0.5071 \\ 0.0461 & -0.8564 \\ 0.1110 & -0.0828 \end{bmatrix}, \quad V_1 = \begin{bmatrix} 0.8917 & 0.4210 \\ 0.0895 & 0.1558 \\ 0.0126 & -0.1709 \\ 0.4429 & -0.8674 \\ 0.0231 & -0.1302 \end{bmatrix}$$

The final reduced order periodic model is of the form of (2.17)-(2.18) with

$$\hat{A}_0 = \begin{bmatrix} 0.7831 & 0.3164 \\ 0.4047 & -0.6883 \end{bmatrix}, \quad \hat{B}_0 = \begin{bmatrix} 0.9185 & 0.8940 \\ 0.5350 & -1.7307 \end{bmatrix},$$

$$\hat{C}_0 = \begin{bmatrix} 0.3883 & -1.0775 \\ -0.0087 & -0.3967 \end{bmatrix}, \quad D_0 = \begin{bmatrix} 0.5000 & 0.0000 \\ 0.0000 & 0.5000 \end{bmatrix},$$

$$\hat{A}_1 = \begin{bmatrix} -0.2993 & -0.4375 \\ -0.5296 & 0.4256 \end{bmatrix}, \quad \hat{B}_1 = \begin{bmatrix} 1.4466 & 0.1062 \\ 0.1804 & -0.3599 \end{bmatrix},$$

$$\hat{C}_1 = \begin{bmatrix} 0.3388 & 0.1372 \\ 1.1311 & 0.0185 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0.6000 & 0.000 \\ 0.000 & 0.6000 \end{bmatrix}$$

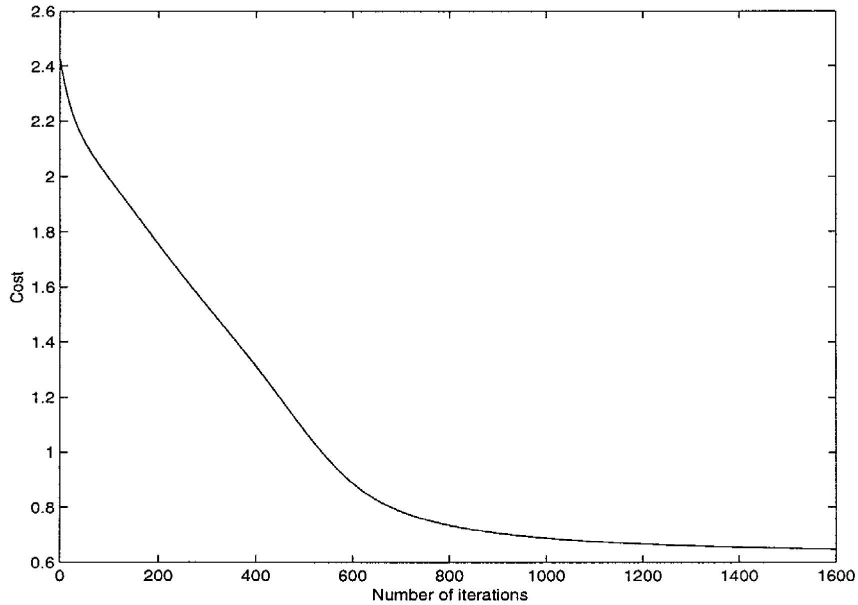

Figure 2. The cost evolution of Example 2.

The above two examples clearly demonstrate the effectiveness of the proposed periodic model reduction algorithms. It should be noted that our algorithm only gives a local minimum solution. The gap between the resulting local minimum and the global minimum depends on the choice of the initial condition. As a sensible choice, one may start from a reduced order model given by the Hankel-norm aproximation in [28].

6. Conclusion

In this paper, analysis and computational algorithms are presented for the H_2 optimal order reduction for linear periodic discrete time systems and digital filters. The order of the discrete time system or digital filter is allowed to be time-varying. By converting the periodic model reduction problem into an unconstrained optimization problem over the Stiefel manifold, two convergent algorithms based on the continuous and discrete gradient flows have been developed. It has been shown that both algorithms guarantee the cost function to decrease monotonically and, hence, converge to the optimal value. The optimality and convergence properties are demonstrated by two numerical examples.

Appendix (Proof of Theorem 1)

Let $\mathcal{D}$ be the Fréchet derivative operator such that $\mathcal{D}U = \xi$ and $\mathcal{D}V = \eta$. Since P, Q, F_c, G_c, H_c, E_c, $\mathcal{J}_1(U, V)$ and $\mathcal{J}_2(U, V)$ are all differentiable functions of

(U, V), their Fréchet derivatives can be written, respectively, as

$$\mathcal{D}P = \mathcal{D}P(\xi, \eta), \quad \mathcal{D}Q = \mathcal{D}Q(\xi, \eta), \quad \mathcal{D}F_c = \mathcal{D}F_c(\xi, \eta), \\ \mathcal{D}G_c = \mathcal{D}G_c(\xi . \eta), \quad \mathcal{D}H_c = \mathcal{D}H_c(\xi, \eta)$$

$$\mathcal{D}\mathcal{J}_1 = \mathcal{D}\mathcal{J}_1(\xi, \eta), \quad \mathcal{D}\mathcal{J}_2 = \mathcal{D}\mathcal{J}_2(\xi, \eta)$$

Taking the Fréchet derivative of (3.12) yields

$$(\mathcal{D}F_c)PF_c^T + F_cP(\mathcal{D}F_c^T) + F_c(\mathcal{D}P)F_c^T - (\mathcal{D}P) \\ + (\mathcal{D}G_c)G_c^T + G_c(\mathcal{D}G_c^T) = 0 \tag{A.1}$$

Multiplying (A.1) by Q and taking the trace give

$$\mathrm{tr}[2(\mathcal{D}F_c)PF_c^TQ + F_c(\mathcal{D}P)F_c^TQ - (\mathcal{D}P)Q + 2(\mathcal{D}G_c)G_c^TQ] = 0$$

i.e.

$$\mathrm{tr}\left[2(\mathcal{D}F_c)PF_c^TQ + F_c^TQF_c(\mathcal{D}P) - Q(\mathcal{D}P) + 2(\mathcal{D}G_c)G_c^TQ\right] = 0$$

This, together with (3.13), implies that

$$\mathrm{tr}[H_c^TH_c(\mathcal{D}P)] = \mathrm{tr}\left[2(\mathcal{D}F_c)PF_c^TQ + 2(\mathcal{D}G_c)G_c^TQ\right] \tag{A.2}$$

Thus the Fréchet derivative of $\mathcal{J}_1(U, V)$ is given by

$$\begin{aligned}(\mathcal{D}\mathcal{J}_1) &= 2\mathrm{tr}\left[PH_c^T(\mathcal{D}H_c)\right] + \mathrm{tr}\left[H_c^TH_c(\mathcal{D}P)\right] \\ &= 2\mathrm{tr}\left[PH_c^T(\mathcal{D}H_c)\right] + 2\mathrm{tr}\left[(\mathcal{D}F_c)PF_c^TQ + (\mathcal{D}G_c)G_c^TQ\right] \\ &= 2\mathrm{tr}\left[PH_c^T(\mathcal{D}H_c) + (\mathcal{D}F_c)PF_c^TQ + (\mathcal{D}G_c)G_c^TQ\right]\end{aligned} \tag{A.3}$$

It is also simple to show that

$$\mathcal{D}\mathcal{J}_2 = 2\mathrm{tr}\{(\hat{E}_0 - E_0)^T(\mathcal{D}\hat{E}_0)\} \tag{A.4}$$

Recall that

$$\hat{F}_0 = \hat{\Phi}_{N,0} = \hat{\Phi}_{N,j+1}U_j^TA_jV_j\hat{\Phi}_{j,0}, \quad j \in [0, N-1] \tag{A.5}$$

$$\hat{G}_0 = [\hat{G}_{00} \;\; \hat{G}_{01} \;\; \hat{G}_{02} \;\; \cdots \;\; \hat{G}_{0(N-1)}], \tag{A.6}$$

$$\begin{aligned}G_{0j} &= \hat{\Phi}_{N,j+1}\hat{B}_j \\ &= \hat{\Phi}_{N,k+1}U_k^TA_kV_k\hat{\Phi}_{k,j+1}\hat{B}_j, \\ &\quad j \in [0, N-1], \; k \in [j+1, N-1]\end{aligned} \tag{A.7}$$

$$\hat{H}_0 = \begin{bmatrix} \hat{H}_{00} \\ \hat{H}_{01} \\ \vdots \\ \hat{H}_{0(N-1)} \end{bmatrix}, \tag{A.8}$$

$$\begin{aligned}\hat{H}_{0j} &= C_jV_j\hat{\Phi}_{j,0} = C_jV_j\hat{\Phi}_{j,k+1}U_k^TA_kV_k\hat{\Phi}_{k,0}, \\ &\quad j \in [0, N-1], \; k \in [0, j-1]\end{aligned} \tag{A.9}$$

It follows that

$$\mathcal{D}\hat{F}_0 = \sum_{j=0}^{N-1} \hat{\Phi}_{N,j+1}(\xi_j^T A_j V_j + U_j^T A_j \eta_j)\hat{\Phi}_{j,0} \tag{A.10}$$

$$\begin{aligned}\mathcal{D}\hat{G}_{0j} &= \sum_{k=j+1}^{N-1} \hat{\Phi}_{N,k+1}\left(\xi_k^T A_k V_k + U_k^T A_k \eta_k\right)\hat{\Phi}_{k,j+1}\hat{B}_j \\ &\quad + \hat{\Phi}_{N,j+1}\xi_j^T B_j\end{aligned} \tag{A.11}$$

$$\mathcal{D}\hat{H}_{0j} = C_j \eta_j \hat{\Phi}_{j,0} + \sum_{k=0}^{j-1} C_j V_j \hat{\Phi}_{j,k+1}\left(\xi_k^T A_k V_k + U_k^T A_k \eta_k\right)\hat{\Phi}_{k,0} \tag{A.12}$$

Using (A.10)–(A.12), we obtain

$$\begin{aligned}&\mathrm{tr}[(\mathcal{D}F_c)PF_c^T Q] \\ &= \mathrm{tr}\left\{[0\ \ I]PF_c^T Q \begin{bmatrix} 0 \\ I \end{bmatrix} \sum_{j=0}^{N-1} \hat{\Phi}_{N,j+1}\left(\xi_j^T A_j V_j + U_j^T A_j \eta_j\right)\hat{\Phi}_{j,0}\right\} \\ &= \sum_{j=0}^{N-1} \mathrm{tr}\left\{\hat{\Phi}_{N,j+1}^T [0\ \ I] Q F_c P \begin{bmatrix} 0 \\ I \end{bmatrix} \hat{\Phi}_{j,0}^T V_j^T A_j^T \xi_j\right. \\ &\qquad \left. + \hat{\Phi}_{j,0}[0\ \ I]PF_c^T Q \begin{bmatrix} 0 \\ I \end{bmatrix} \hat{\Phi}_{N,j+1} U_j^T A_j \eta_j\right\} \\ &= \sum_{j=0}^{N-1} \mathrm{tr}\left(R_{1j}^T \xi_k + S_{1j}^T \eta_k\right)\end{aligned}$$

$$\begin{aligned}&\mathrm{tr}\left[(\mathcal{D}G_c)G_c^T Q\right] \\ &\quad = \mathrm{tr}\left(\begin{bmatrix} 0 \\ I \end{bmatrix} \mathcal{D}\hat{G}_0 G_c^T Q\right)\end{aligned}$$

$$= \mathrm{tr}\left(\begin{bmatrix} 0 \\ I \end{bmatrix} \sum_{j=0}^{N-1} \mathcal{D}\hat{G}_{0j} G_{cj}^T Q\right)$$

$$= \operatorname{tr}\left\{ \begin{bmatrix} 0 \\ I \end{bmatrix} \sum_{j=0}^{N-1} \sum_{k=j+1}^{N-1} \hat{\Phi}_{N,k+1} \left(\xi_k^T A_k V_k + U_k^T A_k \eta_k \right) \hat{\Phi}_{k,j+1} \hat{B}_j G_{cj}^T Q \right.$$

$$\left. + \begin{bmatrix} 0 \\ I \end{bmatrix} \sum_{j=0}^{N-1} \hat{\Phi}_{N,j+1} \xi_j^T B_j G_{cj}^T Q \right\}$$

$$= \operatorname{tr}\left\{ \sum_{j=1}^{N-1} \sum_{k=1}^{j} \begin{bmatrix} 0 \\ I \end{bmatrix} \hat{\Phi}_{N,j+1} \left(\xi_j^T A_j V_j + U_j^T A_j \eta_j \right) \hat{\Phi}_{j,k} \hat{B}_{k-1} G_{c(k-1)}^T Q \right.$$

$$\left. + \sum_{j=0}^{N-1} \begin{bmatrix} 0 \\ I \end{bmatrix} \hat{\Phi}_{N,j+1} \xi_j^T B_j G_{cj}^T Q \right\}$$

$$= \sum_{j=0}^{N-1} \operatorname{tr} \left(R_{2j}^T \xi_j + S_{2j}^T \eta_j \right)$$

$$\operatorname{tr}\left[P H_c^T \mathcal{D} H_c(X) \right]$$

$$= \operatorname{tr}\left(P H_c^T \mathcal{D} \hat{H}_0 [0 \quad -I] \right)$$

$$= \operatorname{tr}\left(P \sum_{j=0}^{N-1} H_{cj}^T \mathcal{D} \hat{H}_{0j} [0 \quad -I] \right)$$

$$= \operatorname{tr}\left\{ P \sum_{j=0}^{N-1} H_{cj}^T \left[C_j \eta_j \hat{\Phi}_{j,0} \right.\right.$$

$$\left.\left. + \sum_{k=0}^{j-1} C_j V_j \hat{\Phi}_{j,k+1} \left(\xi_k^T A_k V_k + U_k^T A_k \eta_k \right) \hat{\Phi}_{k,0} \right] [0 \quad -I] \right\}$$

$$= \operatorname{tr}\left\{ \sum_{j=0}^{N-2} \sum_{k=j+1}^{N-1} \left(\hat{\Phi}_{k,j+1}^T \hat{C}_k^T H_{ck} P \begin{bmatrix} 0 \\ -I \end{bmatrix} \hat{\Phi}_{j,0}^T V_j^T A_j^T \xi_j \right.\right.$$

$$\left. + \hat{\Phi}_{j,0} [0 \quad -I] P H_{ck}^T \hat{C}_k \hat{\Phi}_{k,j+1} U_j^T A_j \eta_j \right)$$

$$\left. + \sum_{j=0}^{N-1} \hat{\Phi}_{j,0} [0 \quad -I] P H_{cj}^T C_j \eta_j \right\}$$

$$= \sum_{j=0}^{N-1} \operatorname{tr} \left(R_{3j}^T \xi_j + S_{3j}^T \eta_j \right)$$

Substituting these into (A.3) yields

$$\mathcal{D}\mathcal{J}_1 = 2\mathrm{tr}\left\{\sum_{j=1}^{N-1}\sum_{l=1}^{3}\left(R_{lj}^T\xi_j + S_{lj}\eta_j\right)\right\} \tag{A.13}$$

We now proceed to derive $\mathcal{D}\mathcal{J}_2(\xi, \eta)$. Note that

$$\mathcal{J}_2(U, V) = \sum_{k=1}^{N-1}\sum_{j=k}^{N-1}\mathrm{tr}\Big[\Big(C_j\Phi_{j,k}B_{k-1} - \hat{C}_j\hat{\Phi}_{j,k}\hat{B}_{k-1}\Big)^T \Big(C_j\Phi_{j,k}B_{k-1} - \hat{C}_j\hat{\Phi}_{j,k}\hat{B}_{k-1}\Big)\Big]$$

Thus,

$$\begin{aligned}\mathcal{D}\mathcal{J}_2(\xi, \eta) &= 2\sum_{k=1}^{N-1}\sum_{j=k}^{N-1}\mathrm{tr}\left\{\Gamma_{jk}^T\mathcal{D}\left(C_jV_j\hat{\Phi}_{j,k}U_{k-1}^TB_{k-1}\right)\right\}\\ &= 2\sum_{k=1}^{N-1}\left\{\sum_{j=k}^{N-1}\mathrm{tr}\left(\hat{\Phi}_{j,k}\hat{B}_{k-1}\Gamma_{jk}^TC_j\eta_j + \hat{\Phi}_{j,k}^T\hat{C}_j^T\Gamma_{jk}B_{k-1}^T\xi_{k-1}\right) + h_k\right\}\end{aligned} \tag{A.14}$$

where

$$h_k = \sum_{j=k+1}^{N-1}\sum_{l=k}^{j-1}\mathrm{tr}\left\{\Gamma_{jk}^T\hat{C}_j\hat{\Phi}_{j,l+1}\left(\xi_l^TA_lV_l + U_l^TA_l\eta_l\right)\hat{\Phi}_{l,k}\hat{B}_{k-1}\right\} \tag{A.15}$$

By exchanging the summation indices, it can be easily shown that

$$\begin{aligned}h_k &= \sum_{j=k}^{N-2}\sum_{l=j+1}^{N-1}\mathrm{tr}\left\{\Gamma_{lk}^T\hat{C}_l\hat{\Phi}_{l,j+1}\left(\xi_j^TA_jV_j + U_j^TA_j\eta_j\right)\hat{\Phi}_{j,k}\hat{B}_{k-1}\right\}\\ &= \sum_{j=k}^{N-2}\mathrm{tr}\left(Y_{kj}^T\xi_j + Z_{kj}^T\eta_j\right)\end{aligned}$$

and

$$\sum_{k=1}^{N-1}h_k = \sum_{j=1}^{N-2}\sum_{k=1}^{N-1}\mathrm{tr}\left(Y_{kj}^T\xi_j + Z_{kj}^T\eta_j\right) \tag{A.16}$$

where

$$Y_{kj} = \sum_{l=j+1}^{N-1}A_jV_j\hat{\Phi}_{j,k}\hat{B}_{k-1}\Gamma_{lk}^T\hat{C}_l\hat{\Phi}_{l,j+1} \tag{A.17}$$

$$Z_{kj} = \sum_{l=j+1}^{N-1}A_j^TU_j\hat{\Phi}_{l,j+1}^T\hat{C}_l^T\Gamma_{lk}\hat{B}_{k-1}^T\hat{\Phi}_{j,k}^T \tag{A.18}$$

It can be further verified that

$$\sum_{k=1}^{N-1}\sum_{j=k}^{N-1} \mathrm{tr}\left(\hat{\Phi}_{j,k}\hat{B}_{k-1}\Gamma_{jk}^T C_j\eta_j\right) = \sum_{j=1}^{N-1}\sum_{k=1}^{j} \mathrm{tr}\left(\hat{\Phi}_{j,k}\hat{B}_{k-1}\Gamma_{jk}^T C_j\eta_j\right) \tag{A.19}$$

$$\sum_{k=1}^{N-1}\sum_{j=k}^{N-1} \mathrm{tr}\left(\hat{\Phi}_{j,k}^T\hat{C}_j^T\Gamma_{jk}B_{k-1}^T\xi_{k-1}\right) = \sum_{j=0}^{N-2}\sum_{k=j+1}^{N-1} \mathrm{tr}\left(\hat{\Phi}_{k,j+1}^T\hat{C}_k^T\Gamma_{k(j+1)}B_j^T\xi_j\right) \tag{A.20}$$

It follows that

$$\mathcal{D}\mathcal{J}_2(\xi,\eta) = 2\sum_{j=0}^{N-1}\mathrm{tr}\left(R_{4j}^T\xi_j + S_{4j}^T\eta_j\right)$$

This, together with (A.13) yields

$$\mathcal{D}\mathcal{J}(\xi,\eta) = 2\mathrm{tr}\left\{\sum_{j=0}^{N-1}\left(R_j^T\xi_j + S_j\eta_j\right)\right\} \tag{A.21}$$

where $R_j = R_{1j} + R_{2j} + R_{3j} + R_{4j}$ and $S_j = S_{1j} + S_{2j} + S_{3j} + S_{4j}$.

The gradient $\nabla\mathcal{J}_V(U,V)$ as defined in (3.16) satisfies the following conditions.

$$\nabla\mathcal{J}(U,V) \in T_{(U,V)}\Psi, \ \ \forall(U,V)\in\Psi \tag{A.22}$$

$$\mathcal{D}\mathcal{J}(\xi,\eta) =< \nabla\mathcal{J}(U,V),(\xi,\eta) >, \ \ \forall(\xi,\eta)\in T_{(U,V)}\Psi \tag{A.23}$$

In view of (A.21), condition (A.23) is equivalent to

$$\sum_{j=0}^{N-1}\left[\nabla\mathcal{J}_{U_j}(U,V) - 2R_j\right]^T\xi_j + \sum_{j=0}^{N-1}\left[\nabla\mathcal{J}_{V_j}(U,V) - 2S_j\right]^T\eta_j = 0, \quad \forall(\xi,\eta)\in T_{(U,V)}\Psi \tag{A.24}$$

Further, it can be easily verified that

$$T_{(U,V)}\Psi^{\perp} = (U_0\Omega_0, U_1\Omega_1, \cdots, U_{(N-1)}\Omega_{N-1}; V_0\Theta_0, V_1\Theta_1, \cdots, V_{N-1}\Theta_{N-1})$$

where $\Omega_j \in \mathcal{R}^{\hat{n}_{j+1}\times\hat{n}_{j+1}}$, $\Theta_j \in \mathcal{R}^{\hat{n}_j\times\hat{n}_j}$ and $\Omega_j = \Omega_j^T,\ \Theta_j = \Theta_j^T,\ j = 0, 1, \cdots, N-1$. This, together with (A.24) implies

$$\nabla\mathcal{J}_{U_j}(U,V) = 2R_j - U_j\Omega_j, \ \ \nabla\mathcal{J}_{V_j}(U,V) = 2S_j - V_j\Theta_j$$

Condition (A.22) uniquely determines that

$$\Omega_j = R_j^T U_j + U_j^T R_j, \ \ \Theta_j = S_j^T V_j + V_j^T S_j$$

Hence Theorem 1 is established.

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